ANALYSIS I MIDTERM EXAMINATION

Total marks: 35

(1) Prove that the Principle of Mathematical Induction, the Principle of Strong Induction and the Well Ordering Property of $\mathbb{N}$ are all equivalent. (6 marks)

(2) Prove that if A_n is a countable set for each $n \in \mathbb{N}$, then their union $\cup_{n=1}^{\infty} A_n$ is countable. Prove that if $B_1, \cdots, B_r$ are a finite number of countable sets, then their product $\prod_{i=1}^{r} B_i$ is countable. (3+3 marks)

(3) State the completeness property of $\mathbb{R}$. Using the completeness property of $\mathbb{R}$, prove that there exists a positive real number x such that $x^2 = 5$. Prove that the ordered field $\mathbb{Q}$ of rational numbers does not satisfy the completeness property. (2+3+1 marks)

(4) (a) Prove that a Cauchy sequence of real numbers converges. (3 marks)

    (b) Compute $lim_{n \to \infty}(n^{1/n})$. (3 marks)

(5) Let $X = (x_n)$ be a sequence of real numbers which converges. Let $lim_{n \to \infty}(x_n) = x$, where $x \in \mathbb{R}$. Let $Y = (y_n)$ be another sequence of real numbers. Prove that $limsup_{n \to \infty}(x_n + y_n) = x + limsup_{n \to \infty}(y_n)$. (5 marks)

(6) Show that the sequence of real numbers $x_n = \sum_{i=1}^{n}(-1)^{i+1}/i$ converges. (6 marks)

Date: September 3, 2015.